

The Homotopy Perturbation Method for Solving the Kuramoto – Sivashinsky Equation

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Abstract: - The approximate solutions for the Kuramoto –Sivashinsky Equation are obtained by using the homotopy perturbation method (HPM). The numerical example show that the approximate solution comparing with the exact solution is accurate and effective and suitable for this kind of problem.

Keywords: - Homotopy Perturbation Method (HPM); Kuramoto –Sivashinsky Equation .

I. INTRODUCTION

The homotopy perturbation method (HPM) was first proposed by the Chinese mathematician Ji-Huan He [1-3]. Unlike classical techniques, the homotopy perturbation method leads to an analytical approximate and exact solutions of the nonlinear equations easily and elegantly without transforming the equation or linearizing the problem and with high accuracy, minimal calculation and avoidance of physically unrealistic assumptions. As a numerical tool, the method provide us with numerical solution without discretization of the given equation, and therefore, it is not effected by computation round-off errors and one is not faced with necessity of large computer memory and time. This technique has been employed to solve a large variety of linear and nonlinear problems [4- 10]. In the present study, homotopy perturbation method has been applied to solve the Kuramoto–Sivashinsky equations. The numerical results are compared with the exact solutions. It is shown that the errors are very small.

II. INDENTATIONS AND EQUATIONS

II.1 Mathematical Model

The Kuramoto–Sivashinsky equation is a non-linear evolution equation and has many applications in a variety of physical phenomena such as reaction diffusion systems (Kuramoto and Tsuzuki, 1976)[11], long waves on the interface between two viscous fluids (Hooper and Grimshaw, 1985)[12], and thin hydrodynamics films (Sivashinsky, 1983)[13]. The Kuramoto-Sivashinsky equation has been studied numerically by many authors (Akrivis and Smyrlis, 2004; Manickam et al.,1998; Uddin et al., 2009)[14-16].

Consider the Kuramoto–Sivashinsky equation

$$u_t + uu_x + \alpha u_{xx} + \gamma u_{xxx} + \beta u_{xxxx} = 0 \quad (1)$$

Subject to the initial condition

$$u(x, 0) = f(x) \quad a \leq x \leq b. \quad (2)$$

And boundary conditions

$$\left. \begin{aligned} u(a, t) = g_1(t), \quad u(b, t) = g_2(t) \quad t > 0 \\ \frac{\partial^2 u}{\partial x^2} = h_1, \quad \text{at } x = a \text{ and } x = b \quad \text{where } h_1 \geq 0. \end{aligned} \right\} \quad (3)$$

II.2 Basic idea of homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following non-linear differential equation

$$A(u) - f(r) = 0, \quad r \in \delta \quad (4)$$

with the following boundary conditions

$$B \left(u, \frac{\partial u}{\partial n} \right) = 0, \quad r \in \tau \quad (5)$$

where A is a general differential operator, B a boundary operator, $f(r)$ is a known analytical function and τ is the boundary of the domain δ . The operator A can be decomposed into two operators, L and N , where L is a linear, and N a nonlinear operator. Eq. (4) can be, therefore, written as follows:

$$L(u) + N(u) - f(r) = 0. \quad (6)$$

Using the homotopy technique, we construct a homotopy $v(r, p) : \delta \times [0, 1] \rightarrow R$, which satisfies:

$$\begin{aligned} H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \\ p \in [0, 1], r \in \delta. \end{aligned} \quad (7)$$

Or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \quad (8)$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation for the solution of Eq. (4), which satisfies the boundary conditions. Obviously, from Eqs. (7) and (8) we will have:

$$H(v, 0) = L(v) - L(u_0) = 0, \quad (9)$$

$$H(v, 1) = A(v) - f(r) = 0. \quad (10)$$

The changing process of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called homotopy. According to the (HPM), we can first use the embedding parameter p as a small parameter, and assume that the solution of Eqs. (7) and (8) can be written as a power series in p :

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (11)$$

Setting $p = 1$, results in the approximate solution of Eq. (1)

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (12)$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques.

The series (11) is convergent for most cases. Some criteria is suggested for convergence of the series (11), in our equation, in[1].

II.3 Derivative of HPM for Kuramoto-Sivashinsky equation

consider Kuramoto-Sivashinsky equation when

$$u_t + uu_x + \alpha u_{xx} + \gamma u_{xxx} + \beta u_{xxxx} = 0, \quad (13)$$

with the initial condition of

$$u(x, 0) = f(x) \quad (14)$$

To solve Eq. (13) by means of HPM, we construct the following homotopy for this equation:

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad (15)$$

where $L(v) = \frac{\partial v}{\partial t}$, $L(u_0) = \frac{\partial u_0}{\partial t}$,

that is

$$H(v, p) = (1 - p) \left[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} \right] + p \left[\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \alpha \frac{\partial^2 v}{\partial x^2} + \gamma \frac{\partial^3 v}{\partial x^3} + \beta \frac{\partial^4 v}{\partial x^4} \right] = 0, \quad (16)$$

Substituting v from Eq. (8) into Eq. (13) and equating the terms with identical powers of p , we have

$$p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \quad v_0(x, 0) = f(x); \quad (17)$$

$$p^1 : \frac{\partial v_1}{\partial t} + \frac{\partial v_0}{\partial t} + v_0 \frac{\partial v_0}{\partial x} + \alpha \frac{\partial^2 v_0}{\partial x^2} + \gamma \frac{\partial^3 v_0}{\partial x^3} + \beta \frac{\partial^4 v_0}{\partial x^4} = 0, \quad v_1(x, 0) = 0; \quad (18)$$

$$p^2 : \frac{\partial v_2}{\partial t} + v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x} + \alpha \frac{\partial^2 v_1}{\partial x^2} + \gamma \frac{\partial^3 v_1}{\partial x^3} + \beta \frac{\partial^4 v_1}{\partial x^4} = 0, \quad v_2(x, 0) = 0; \quad (19)$$

$$p^3 : \frac{\partial v_3}{\partial t} + v_0 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_0}{\partial x} + v_1 \frac{\partial v_1}{\partial x} + \alpha \frac{\partial^2 v_2}{\partial x^2} + \gamma \frac{\partial^3 v_2}{\partial x^3} + \beta \frac{\partial^4 v_2}{\partial x^4} = 0, \quad v_3(x, 0) = 0; \quad (20)$$

...

Solving these equations yields v_0, v_1, v_2, v_3 and so on.

Thus, we can obtain

$$u = \sum_{i=0}^n v_i = v_0 + v_1 + v_2 + \dots + v_n \quad (21)$$

III. FIGURES AND TABLES

III.1 Numerical Example

In this section, we apply the technique discussed in the previous section to find numerical solution of the Kuramoto-Sivashinsky equations and compare our results with exact solutions.

Example: [17]

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0, \quad x \in [0, 32\pi], \quad t \in [0, 001]; \quad (22)$$

with the initial condition of

$$u(x, 0) = \cos\left(\frac{x}{16}\right) \left(1 + \sin\frac{x}{16}\right); \quad (23)$$

Exact solution of problem is given by

$$u(x, t) = \cos\left(\frac{x}{16} - t\right) \left(1 + \sin\left(\frac{x}{16} - t\right)\right) \quad (24)$$

$$p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \quad v_0(x, 0) = \cos\left(\frac{x}{16}\right) \left(1 + \sin\frac{x}{16}\right); \quad (25)$$

$$p^1 : \frac{\partial v_1}{\partial t} + \frac{\partial v_0}{\partial t} + v_0 \frac{\partial v_0}{\partial x} + \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^4 v_0}{\partial x^4} = 0, \quad v_1(x, 0) = 0; \quad (26)$$

$$p^2 : \frac{\partial v_2}{\partial t} + v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x} + \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^4 v_1}{\partial x^4} = 0, \quad v_2(x, 0) = 0; \quad (27)$$

$$p^3 : \frac{\partial v_3}{\partial t} + v_0 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_0}{\partial x} + v_1 \frac{\partial v_1}{\partial x} + \frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^4 v_2}{\partial x^4} = 0, \quad v_3(x, 0) = 0; \quad (28)$$

...
Then, we only find third-order term approximate solution for Eq.(22)

$$v_0(x, t) = \cos\left(\frac{x}{16}\right) \left(1 + \sin\left(\frac{x}{16}\right)\right); \quad (29)$$

$$v_1(x, t) = t \cos\left(\frac{x}{16}\right) \left(\frac{9200 \sin\left(\frac{x}{16}\right) - 8192 \cos^2\left(\frac{x}{16}\right) \sin\left(\frac{x}{16}\right) - 12288 \cos^2\left(\frac{x}{16}\right) + 8447}{65536} \right) \quad (30)$$

$$v_2(x, t) = t^2 \cos\left(\frac{x}{16}\right) \left(\frac{165437696 \sin\left(\frac{x}{16}\right) - 517734400 \cos^2\left(\frac{x}{16}\right) \sin\left(\frac{x}{16}\right) + 201326592 \cos^4\left(\frac{x}{16}\right) \sin\left(\frac{x}{16}\right)}{8589934592} + \frac{-579706880 \cos^2\left(\frac{x}{16}\right) + 419430400 \cos^4\left(\frac{x}{16}\right) + 164855297}{8589934592} \right) \quad (31)$$

$$v_3(x, t) = t^3 \cos\left(\frac{x}{16}\right) \left(\frac{4859359014912 \sin\left(\frac{x}{16}\right) - 30758081134592 \cos^2\left(\frac{x}{16}\right) \sin\left(\frac{x}{16}\right)}{1688849860263936} + \frac{38638599536640 \cos^4\left(\frac{x}{16}\right) \sin\left(\frac{x}{16}\right) - 8796093022208 \cos^6\left(\frac{x}{16}\right) \sin\left(\frac{x}{16}\right) - 33071821131776 \cos^2\left(\frac{x}{16}\right)}{1688849860263936} + \frac{52566188621824 \cos^4\left(\frac{x}{16}\right) - 23570780520448 \cos^6\left(\frac{x}{16}\right) + 4826353967871}{1688849860263936} \right) \quad (32)$$

Then approximation solution is $u(x, t) = u_0 + u_1 + u_2 + u_3$ with third-order approximation.

Now we compare exact solution with homotopy perturbation method (HPM) solution in Fig.1, Fig.2.

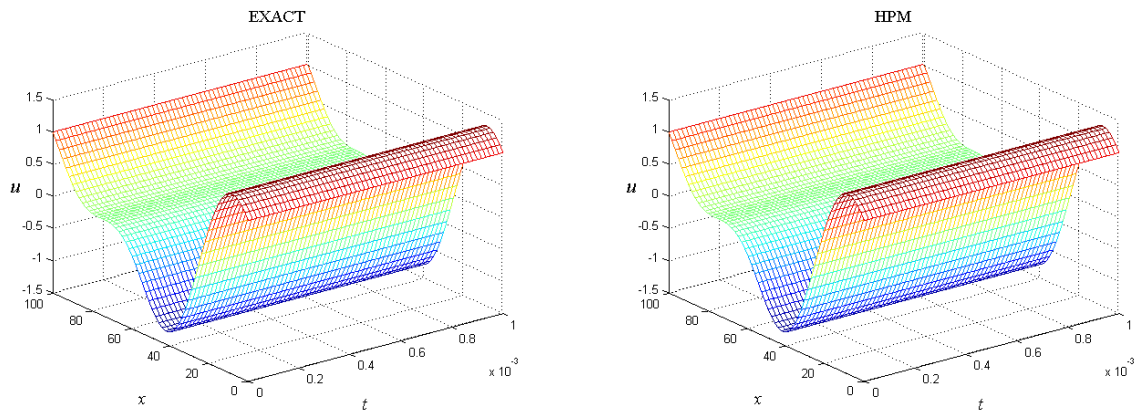


Table (1) comparison exact with homotopy perturbation method (HPM)

$x * \pi$	t	u_{Exact}	u_{HPM}	$ u_{Exact} - u_{HPM} $
0	0	1.000000000000000	1.000000000000000	0
	0.0002	0.999799980005333	0.999988278219568	1.882982142341616e-004
	0.0004	0.999599920042668	0.999976556481800	3.766364391323274e-004
	0.0006	0.999399820144005	0.999964834786718	5.650146427130798e-004
	0.0008	0.999199680341350	0.999953113134345	7.534327929941131e-004
	0.001	0.998999500666708	0.999941391524700	9.418908579912344e-004
6.4	0	0.602909620521184	0.602909620521184	0
	0.0002	0.603261605525986	0.602924029951888	3.375755740976372e-004
	0.0004	0.603613531113795	0.602938440031289	6.750910825057410e-004
	0.0006	0.603965397251127	0.602952850759426	1.012546491701349e-003
	0.0008	0.604317203904505	0.602967262136334	1.349941768170049e-003
	0.001	0.604668951040459	0.602981674162053	1.687276878405974e-003
12.8	0	-1.284545252522524	-1.284545252522524	0
	0.0002	-1.284489444647476	-1.284551820710811	6.237606333447943e-005
	0.0004	-1.284433528322049	-1.284558388250076	1.248599280267992e-004
	0.0006	-1.284377503541076	-1.284564955140284	1.874515992073000e-004
	0.0008	-1.284321370299405	-1.284571521381397	2.501510819914454e-004
	0.001	-1.284265128591897	-1.284578086973377	3.129583814804882e-004
19.2	0	-0.333488736227371	-0.333488736227371	0
	0.0002	-0.333668118536193	-0.333484164639820	1.839538963727683e-004
	0.0004	-0.333847544554259	-0.333479593175463	3.679513787964717e-004
	0.0006	-0.334027014266963	-0.333475021834294	5.519924326690129e-004
	0.0008	-0.334206527659685	-0.333470450616308	7.360770433769148e-004
	0.001	-0.334386084717798	-0.333465879521502	9.202051962963198e-004
25.6	0	0.015124368228711	0.015124368228711	0
	0.0002	0.015095977652350	0.015123677789543	2.770013719346869e-005
	0.0004	0.015067621719845	0.015122987353990	5.536563414442614e-005
	0.0006	0.015039300412906	0.015122296922050	8.299650914345494e-005
	0.0008	0.015011013713235	0.015121606493723	1.105927804877852e-004
	0.001	0.014982761602528	0.015120916069009	1.381544664819934e-004
32	0	1.000000000000000	1.000000000000000	0
	0.0002	0.999799980005333	0.999988278219567	1.882982142346057e-004
	0.0004	0.999599920042668	0.999976556481800	3.766364391322163e-004
	0.0006	0.999399820144005	0.999964834786718	5.650146427135239e-004
	0.0008	0.999199680341350	0.999953113134344	7.534327929941131e-004
	0.001	0.998999500666708	0.999941391524699	9.418908579915675e-004

IV. CONCLUSION

The (HPM) applied to Kuramoto-Sivashinsky equation and by comparing with the exact solution **Fig.(1)**, **Fig.(2)** and **Table(1)** shows that the absolute error is so small and the approximate solution is so closed to the exact solution.

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